Indian J.Sci.Res. 11 (1): 093-099, 2015
ISSN: 0976-2876 (Print)
ISSN: 2250-0138(Online)

GAP FUNCTION ON VARIATIONAL-LIKE INEQUALITY IN BANACH SPACE

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ABSTRACT

In this paper, we consider and extend the theory of gap function from Extended Variational Inequality (EVI) problems to the case of Extended Variational like Inequality (EVLI) problems in Banach space.

KEYWORD: Variational Inequality, Variational-Like Inequality, Gap Functions

AMS classification: 47H04, 47H06, 47H10, 47J20, 47J25, 47J30, 49J40, 90C

Variational inequality problems theory has emerged as elegant and fascinating branch of applicable mathematics in recent years, because it describes a broad spectrum of interesting and important developments involving a link among various fields of mathematics, physics, economics, engineering, mechanics, etc. In last three decades, variational inequality theory has been extended and generalized in several directions, using new and powerful methods, to study a wide class of problems in a unified and general framework. See for example [4-7]

One of the most important and interesting problems in the theories of vatiational inequality is to develop the methods which give efficient and implementable algorithms for solving variational inequalities problems. These methods include projection method and its variant forms, linear approximation, decent, and Newton's methods, and the method based on auxiliary principle technique.

It is well known that the projection method and its variants cannot be extended for Banach spaces. To overcome this drawback, we use gap function.

Given K closed and convex in $\mathbb{R}^n$ and a function $F : K \to \mathbb{R}^n$, the variational inequality problem (in short *VI*) is to find $x_0 \in K$ such that

$$\left\langle F\left(x_0\right), y - x_0\right\rangle \geq 0 \quad \text{for all } y \in K$$

where $\langle \cdot,\cdot \rangle$ denote the inner product in $\mathbb{R}^n$. Due to its applications to such diverse areas as partial differential equations, mathematical, economics and operations research, *VI* has been investigated by many authors and various methods and algorithms to solve *VI* has been made by introducing the concept of a *gap function* is well known both in the context of convex optimization [8] as well as variational enequality [9]. The minimization of gap functions is a viable approach for solving variational inequalities. A function $\varphi : K \to \mathbb{R} \cup \{+\infty\}$ is called a gap function for *VI*, if

$$(i) \qquad \varphi(x) \geq 0 \qquad \text{for all } x \in K \quad \textit{and}$$
$$(ii) \qquad \varphi(x_0) = 0 \quad \text{ifand only if } x_0 \text{ solves } VI.$$

An equivalent minimization problem is defined for the VI, with a gap function:

$$\text{minimize} \quad \varphi(x) \quad \text{subject to } x \in K.$$

In this paper we generalize the gap function for extended variational inequalities [10] to extended variational – like inequalities in a general Banach space.

PRELIMINARIES

In this section, we study some properties of *invex* functions. Let X and Y are real Banach spaces and let $\eta : K \times K \to K$, where K a subset of X, having the connectedness property: $y + \lambda\eta(x, y) \in K$, and $\eta(x, y) = -\eta(y, x), \forall x, y \in K, \lambda \in [0,1]$. The real function $f : K \to \mathbb{R}$ is a *pre – invex function* with respect to a function η, if

$$f\left(x+\lambda\eta(x,y)\right) \leq (1-\lambda) f(x) - \lambda f(y); \forall x, y \in K, \lambda \in [0,1]$$

The function f is an *invex function* with respect to η, if

$$f(x) - f(y) \geq \left\langle f'(y), \eta(x, y)\right\rangle, \forall\ x, y \in K.$$

Remark (1) If f is a differentiable pre – invex function with respect to η, then f is an invex function with respect to the same η.

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Definition (2.0) The function f is locally Lipschitz on K if, for each point $x \in K$ there exist a neighbourhood $x+\delta B$ and a positive constant θ such that

$$|f(y)-f(z)| \le \theta \|y-z\|, \text{ whenever } y,z \in x+\delta B$$

Definition (2.1)

(i) If $f: K \to \mathbb{R}$ is a local Lipschyiz function on K, then $v \in X^*$ (X^* is the dual of X) is a η-***subgradient*** of f at x if

$$f(y)-f(x) \ge \langle v, \eta(y,x) \rangle, \text{ for all } y \in K.$$

(ii) The set of all η- ***subgradient*** of f at x is called the η - ***subdifferential*** of f at x and is denoted by $\partial^{\eta} f(x)$.

(iii) Let $f: K \to \mathbb{R}$ be a local Lipschyiz function on K, and $x \in K$, then f is *generalized invex* at x with respect to some function η if

$$f(y)-f(x) \ge \langle v, \eta(y,x) \rangle, \text{ for all } v \in \partial^{\eta} f(x), y \in K$$

where $\partial^{\eta} f(x)$ denotes the η - *subdifferential* of f at x as in Clarke [1].

Definition (2.2) If $f: K \to \mathbb{R}$ be a local Lipschyiz function, the η- *generalized directional derivative* of f at x in the direction h with respect to $\eta(x,y)$, denoted by

$$f_{\eta}^{\circ}(x,h) = \begin{cases} -\infty & \text{if } \partial^{\eta} f(x) = \phi \\ \sup\{\langle v,h \rangle : v \in \partial^{\eta} f(x)\} & \text{if } \partial^{\eta} f(x) \ne \phi \end{cases}$$

Condition C (2.3) [2] Let K be a invex subset of X with respect to $\eta: K \times K \to K$; we say that the function η satisfies the condition C if for any x, y and for any $\lambda \in [0,1]$,

$$\eta(y, y+\lambda \eta(x,y)) = -\lambda \eta(x,y)$$
$$\eta(x, y+\lambda \eta(x,y)) = (1-\lambda) \eta(x,y)$$

Theorem (2.1) Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a pre – invex function with respect to η; satisfying condition C. If f is bounded above on an open subset. Then f is locally Lipschitz on the interior of $Dom(f)$.

Proof: (a) Suppose then that f is bounded on a ball $x_0+\delta B \subset Dom(f)$ by a constant $a < +\infty$. For each $x \in X$, we take $x_0 = x+\theta\eta(x,y)$ where $\theta = \dfrac{\|x-x_0\|}{\delta+\|x-x_0\|} < 1$. Then

$$\eta(y,x_0) = \eta(y, x+\eta(y,x)) = (1-\theta)\eta(x,y).$$

Therefore $\|\eta(y,x_0)\| = (1-\theta)\|\eta(x,y)\| = \delta$ and consequently, $f(y) \le a$.

The pre-invexity of f implies that

$$f(x_0) = f(x+\theta\eta(x,y)) \le \theta a + (1-\theta) f(x)$$

$$\Rightarrow (1-\theta) f(x_0) \le \theta(a - f(x_0)) + (1-\theta) f(x)$$

$$\Rightarrow f(x_0) \le \frac{\theta(a-f(x_0))}{1-\theta} + f(x)$$

Therefore

$$f(x_0) - f(x) \le \frac{a-f(x_0)}{\delta}\|x-x_0\|, \quad \text{for all } x \in X$$

Now take $x \in x_0+\delta B$ and $x = x_0+\theta\eta(y,x_0)$, where $\theta = \dfrac{\|x-x_0\|}{\delta} \le 1$. Then $\|\eta(y,x_0)\| \le \delta$ and consequently, $f(x) \le a$. The pre-invexity of f implies that

$$f(x) = f(x_0+\theta\eta(y,x_0)) \le \theta f(y) + (1-\theta) f(x_0) \le \theta a + (1-\theta) f(x_0)$$

implies $f(x) - f(x_0) \le \dfrac{(a-f(x_0))}{\delta}\|x-x_0\|$

From (2.1) and (2.2) implies that

$$\forall\, x \in x_0+\delta B, \quad f(x) - f(x_0) \le \frac{(a-f(x_0))}{\delta}\|x-x_0\|$$

consequently, that f is continuous at x_0.

(b) We now prove that f is Lipchitz on the ball $x_0+\beta B$, where $\beta < \delta$. Fix an integer n larger then

$\frac{\|x_1 - x_0\|}{\delta - \beta}$. Take x_1 and x_2 in $x_0 + \beta B$ and divide the segment from x_1 and x_2 into n parts, using the points $y_i = x_1 + \frac{j}{n}(x_2 - x_1)$, $(j = 0, 1, \ldots, n)$. Note that $y_0 = x_1, y_n = x_2$ and the points $y_i \in x_0 + \beta B$. It is clear that $y_j + (\delta - \beta)B$ are contained in $x_0 + \beta B$, so that f is bounded by a on $y_j + (\delta - \beta)B$. From (2.3), with x_0 replaced by y_j and δ replaced by $\theta - \beta$, implies that

$$\left|f(y_{j+1}) - f(y_j)\right| \le \frac{(a - f(y_j))}{\delta - \theta}\|y_{j+1} - y_j\|$$

Since $\|y_{j+1} - y_j\| = \frac{\|x_1 - x_0\|}{n} \le \delta - \theta$. On the other hand, (2.3) implies that

$$f(x_0) - f(y_j) \le \frac{(a - f(x_0))}{\delta}\|y_j - x_0\| \le a - f(x_0)$$

Then

$$\left|f(x_1) - f(x_2)\right| \le \sum_{j=1}^{n-1}\left|f(y_{j+1}) - f(y_j)\right| \le \frac{2(a - f(x_0))}{\delta - \beta}\|x_1 - x_2\|$$

Thus, f is Lipschitz on $x_0 + \beta B$.

(c) Lastly, we shall prove that f is Lipschitz on a suitable neighbourhood on each point x_1 in the interior of the domain of f. It is sufficient to show that f is bounded above on a neighbourhood of x_1. Let $\gamma > 0$ be such that $x_1 + \gamma B$ is contained in $\mathrm{Dom}(f)$. Set $\lambda = \frac{\gamma}{\lambda + \|x_1 - x_0\|}$, which is strictly less than 1. It is easy to see that the element

$$x_2 = x_0 + \frac{1}{1 - \lambda}\eta(x_1, x_0) \in x_1 + \gamma B$$

and that f is bounded above on the ball $x_1 + \lambda\delta B$, then the element z such that $y = x_2 + \lambda\eta(z, x_2) \in x_0 + \delta B$. Then $f(z) \le a$ and by pre-invexity,

$$f(y) = f(x_2 + \lambda\eta(z, x_2)) \le \lambda f(z) + (1 - \lambda)f(x_2) \le \lambda a + (1 - \lambda)f(x_2)$$

This complete the proof of theorem.

Theorem (2.2) Let U be an open invex subset of X, with η and let $f : U \to \mathbb{R}$ is pre-invex and bounded above function with the same η, satisfying the condition C. Then for any x in U, for $\theta > 0$ and $\delta > 0$ we have

$$\left|f(x_1) - f(x_2)\right| \le \theta\|\eta(x_1, x_2)\| \quad \text{for all } x_1, x_2 \in x + \delta B$$

Proof: Since from the last part (c) of Theorem (2.2), f is bounded on $x + \delta B$. Now, let M be the bound on $|f|$ on the set $x + 2\delta B$, where $\delta > 0$. For distinct $x_1, x_2 \in x + \delta B$, set

$$x_3 = \left(\frac{\delta}{\alpha}\right)\eta(x_2, x_1) \quad \text{where} \quad \alpha = \|\eta(x_2, x_1)\|.$$

Since $x + \delta B$ is invex, therefore $x_3 \in x + 2\delta B$ implies $x_2 = x_3 + \left(\frac{\delta}{\alpha}\right)\eta(x_1, x_2)$ implies $x_2 \in x + 2\delta B$. We suppose $x_2 = x_1 + \left(\frac{\delta}{\alpha}\right)\eta(x_3, x_1)$ implies

$$x_2 = x_3 + \left(\frac{\delta}{\alpha}\right)\eta\left(x_1, x_1 + \left(\frac{\alpha}{\alpha + \delta}\right)\eta(x_3, x_1)\right)$$

$$= x_3 + \left(\frac{\delta}{\alpha}\right)\left[-\left(\frac{\alpha}{\alpha + \delta}\right)\eta(x_3, x_1)\right]$$

$$= x_3 + \left(\frac{\delta}{\alpha + \delta}\right)\eta(x_1, x_3)$$

By the pre-invexity of f we have

$$f(x_2) \le \left(\frac{\delta}{\alpha + \delta}\right)f(x_1) + \left(1 - \frac{\delta}{\alpha + \delta}\right)f(x_3) = \left(\frac{\delta}{\alpha + \delta}\right)f(x_1) + \left(\frac{\alpha}{\alpha + \delta}\right)f(x_3)$$

Then

$$f(x_2) - f(x_1) \le \left(\frac{\delta}{\alpha + \delta}\right)\left[f(x_3) - f(x_1)\right] \le \left(\frac{\alpha}{\delta}\right)\left[f(x_3) - f(x_1)\right].$$

Since $|f| \le M$ and $\alpha = \|\eta(x_2, x_1)\|$, yields

$$f(x_2) - f(x_1) \le \frac{2M}{\delta}\|\eta(x_2, x_1)\|.$$

Since x_1 and x_2 are arbitrary may be interchanged, we have conclude that

$$\left|f(x_2)-f(x_1)\right|\le\theta\left\|\eta(x_2,x_1)\right\|,\ \text{for all}\ x_1,x_2\in x+\delta B,\theta>0$$

.

Theorem (2.2) Let function $f:K\to\mathbb{R}$ be a local Lipschitz function on K and $x_0\in K$. If f is generalized invex at x_0 with respect to some function η, then

$$\partial f(x_0)\subset\partial^{\eta}f(x_0).$$

Hence $\partial^{\eta}f(x_0)$ is nonempty closed convex set.

Proof: Since f is a local Lipschitz function on K, then $\partial f(x_0)$ is non-empty bounded closed convex set by Clarke [1, Prop.(2.1.2)]. Let $v\in\partial f(x_0)$. Therefore

$$f(x)-f(x_0)\ge\langle v,\eta(x,x_0)\rangle,\ \text{for all}\ x\in K\quad .$$

From the definition of η- subgradient, we have $v\in\partial^{\eta}f(x_0)$. That complete the proof.

Proposition (2.4) When f is invex on $U\subseteq X$ with respect to η, satisfies the condition C, and near x we have

$$\left|f(y)-f(z)\right|\le\theta\left\|\eta(y,z)\right\|,\ \text{for all}\ y,z\in x+\delta B$$

(i) Then $f_{\eta}^{\circ}(x,v)$ concides with the directional derivative $f_{\eta}'(x,v)$ for each *v*.

(ii) If f is pre-invex with respect to some η and locally Lipschitz at *x*, $\partial^{\eta}f(x)$ coincides with the η-subdifferential of f at x in the sense of convex analysis.

Proof: (i) Suppose $x\in\operatorname{int}Dom(f)$. Since for $\theta>0,\ \delta>0$

$$\left|f(y)-f(z)\right|\le\theta\left\|\eta(y,z)\right\|\ \text{for all}\ y,z\in x+\delta B$$

Then for all $\alpha\le\frac{\delta}{2}$ and $\beta\le\frac{\delta}{2\|\eta(z,y)\|}$,

$$-\theta\eta(z,y)\le\frac{f(y+t\eta(z,y))-f(y)}{t}\le\theta\eta(z,y)$$

when $y\in x+\alpha B$ and $t\le\beta$. It follows that

$$-\theta\eta(z,y)\le f_{\eta}^{\circ}(x,\eta(z,y))=\inf_{\alpha,\beta>0}\sup_{\|\eta(z,y)\|\le\alpha}\frac{f(y+t\eta(z,y))-f(y)}{t}\le\theta\eta(z,y)$$

Whence, f is Clarke differentiable; in particular, we obtain the inequality

$$\left|f_{\eta}^{\circ}(x,\eta(z,y))\right|\le\theta\left\|\eta(z,y)\right\|\ \Rightarrow\ f_{\eta}^{\circ}(x,.)\quad\text{is}$$

finite.

Since

$$f_{\eta}^{\circ}(x,\eta(z,y))=\lim_{t\to 0+}\sup_{\|\eta(z,y)\|\le\varepsilon\delta}\ \sup_{0<t<\varepsilon}\frac{f(y+t\eta(z,y))-f(y)}{t}$$

where δ is any fixed positive number. We show that by definition of pre-invex function that the function

$$t\to\frac{f(y+t\eta(z,y))-f(y)}{t}\quad\text{is non – decreasing.}$$

In fact, if $t_1\le t_2$ then

$$f(y+t_1\eta(z,y))-f(y)=f\left(y+\tfrac{t_1}{t_2}\eta(y+t_2\eta(z,y),y)\right)-f(y)$$

$$\le\left(1-\frac{t_1}{t_2}\right)f(y)+\frac{t_1}{t_2}f(y+t_2\eta(x,y))-f(y)$$

$$=\frac{t_1}{t_2}\left[f(y+t_2\eta(z,y))-f(y)\right]\quad(*)$$

Therefore

$$\frac{f(y+t_1\eta(z,y))-f(y)}{t_1}\le\frac{f(y+t_2\eta(z,y))-f(y)}{t_2}$$

Thus $t\to\dfrac{f(y+t\eta(z,y))-f(y)}{t}$ is non – decreasing, whence

$$f_{\eta}^{\circ}(x,\eta(z,x))=\lim_{\varepsilon\to 0+}\sup_{\|\eta(x,y)\|\le\varepsilon\delta}\frac{f(y+\varepsilon\eta(z,y))-f(y)}{\varepsilon}$$

Now by the condition (*), for any y in $x'+\varepsilon\delta B$ one has,

$$\left|\frac{f(y+\varepsilon\eta(z,y))-f(y)}{\varepsilon}-\frac{f(x+\varepsilon\eta(z,x))-f(x)}{\varepsilon}\right|$$

$$\leq \left| \frac{f(y+\varepsilon\eta(z,y))-f(y)}{\varepsilon} \right| + \left| \frac{f(x+\varepsilon\eta(z,x))-f(x)}{\varepsilon} \right|$$

$$\leq \theta\|\eta(z,y)\| + \theta\|\eta(z,x)\|$$
$$\leq 2\delta\theta$$

So that

$$f_\eta^\circ(x,\eta(z,x)) \leq \lim_{\varepsilon\to 0+} \frac{f(x+\varepsilon\eta(z,x))-f(x)}{\varepsilon} + 2\delta\theta \leq f_\eta'(x,\eta(z,x)) + 2\delta\theta$$

Since δ is arbitrary, we deduce

$$f_\eta^\circ(x,\eta(z,x)) \leq f_\eta'(x,\eta(z,x))$$

Also by definition

$$f_\eta'(x,\eta(z,x)) \leq f_\eta^\circ(x,\eta(z,x))$$

Hence $\quad f_\eta^\circ(x,\eta(z,x)) = f_\eta'(x,\eta(z,x))$.

(ii) The proof is straight forward from (i).

Proposition (2.5) Let $f : K \to \mathbb{R}$ be invex with η such that η is an open map and $\eta(x,x)=0$ for all $x \in K$. Let $x_0 \in K$. If there exists a neighbourhood U of x_0, such that

$$|f(y)-f(x)| \leq \theta\|\eta(y,x)\|$$

where $\theta > 0$, then $\partial^\eta f(x)$ is weak* - compact.

Proof of the result is on the similar lines of Proposition (2.2)[3].

Remark (2) From the above result it follows that if f is invex with an arbitrary η such that $f_\eta^\circ(x,h)$ is finite for all h, then it is easy to see that $\partial^\eta f(x)$ is compact if and only if

$$|f_\eta^\circ(x,h)| \leq \theta\|h\|, \quad \text{for all } h \in K$$

Mohan and Neogi [2] showed that if $f : K \to \mathbb{R}$ is invex with a given η satisfying Condition C, then f is pre-invex with the same η.

Definition (2.3) Let $K \subset X$ be invex with η. Then η-normal cone $N_K^\eta(x)$ at $x \in K$ is defined as

$$N_K^\eta(x) = \{v \in X^* : \langle v, \eta(y,x)\rangle \leq 0, \text{ for all } y \in K\}$$

Remark (3) It is clear that $0 \in N_K^\eta(x)$.

Proposition (2.6) Let $K \subset X$ be invex with η and $f : K \to \mathbb{R}$ be a differentiable invex with the same η satisfying condition C. Let $v \in X^*$. If

$$v - \nabla f(x) \in N_K^\eta(x)$$

then $v \in \partial^\eta f(x)$.

Proof of the above proposition is on the similar lines as in [3].

Remark (4) Note that by the above proposition $\nabla f(x) \in \partial^\eta f(x)$, since

$$\nabla f(x) - \nabla f(x) = 0 \in N_K^\eta(x)$$

OPTIMISITION WITH INVEX FUNCTIONS

If $f : K \to \mathbb{R}$ is invex at x_0, a global minimum of f, then $0 \in \partial^\eta f(x_0)$ is both a necessary and sufficient condition for optimality. But if x_0 is a local mininum, then it is not necessary that $0 \in \partial^\eta f(x_0)$. (see, example (3.1) [3]). Now we shall show that under certain assumptions the condition that $0 \in \partial^\eta f(x)$ where x is a local minimum of f become necessary.

Penot [4] has introduced the notion of a lower subdifferential $\partial_l f(x)$ for a function $f : K \to \mathbb{R}$ at x as follows

$$\partial_l f(x) = \{v \in X^* : f'(x,h) \geq \langle v,h\rangle \text{ for all } h \in K\}$$

where $f'(x,h) = \lim_{t\to 0+} \frac{f(x+th)-f(x)}{t}$ assumed to exist.

If $x \in K \subset X$ is a local minimum then $f'(x,h) \geq 0$ for all $h \in K$ and hence the condition that $0 \in \partial_l f(x)$ turns out to be necessary condition for a local minimum. We futher note that when f is convex, $\partial_l f(x)$ coincides with the subdifferential of convex analysis [ ] Following Penot, Dutta [3] introduce the lower η- subdifferential of f with to a given η as follows:

$$\partial_l^\eta f(x)=\{v\in X^*: f'(x,\eta(y,x))\geq\langle x,\eta(y,x)\rangle \text{ for all } y\in K\}$$

By definition, if x is a local minimum of f, then $0\in\partial_l^\eta f(x)$. Thus at a local minimum $\partial_l^\eta f(x)\neq\phi$.

Proposition (3.1) Let K be invex with respect to η, and let $f:K\to\mathbb{R}$ be a invex function with respect to the same η, satisfying condition C. Let $x\in K$ be a local minimum of f. Suppose that $f'(x,\eta(y,x))$ exists for all $y\in K$. Then $0\in\partial_l^\eta f(x)$.

Proof of the theorem is on the similar lines of the proof of the Theorem (3.1) of Dutta et al. [3].

Proposition (3.2) Let $K\subset X$ be invex with η. If $f,g:K\to\mathbb{R}$ be Lipschitz and and generalized invex with respect to the same η, satisfying condition C, so is $f+g$ and for each $x\in K$.

$$\partial_l^\eta(f+g)(x)\subseteq\partial_l^\eta f(x)+\partial_l^\eta g(x)$$

The proof is straight forward and omitted.

Definition (3.1) (Distance Function) Let K be a nonempty subset of X, and consider its distance function with η; that is, the function $d_K^\eta(\cdot):X\to\mathbb{R}$ defined by

$$d_K^\eta(x)=\inf\{\|\eta(x,c)\|:c\in K\}$$

If K happen to be closed, then $x\in K$ if and only if $d_K^\eta(x)=0$. Clarke [1] showed that

$$N_l^\eta=cl\left\{\bigcup_{\lambda\geq0}\lambda d_K^\eta(x)\right\}$$

where cl denotes weak* - closure.

Proposition (3.3)[1] If f is locally Lipschitz near x, K is a invex subset of X and attains a minimum over K at x with respect to η, satisfying condition C. Then

$$0\in\partial_l^\eta f(x)+N_K^\eta(x).$$

Proof: see Clarke [1, Corollary and Proposition 2.44].

Extended Variational – like Inequality (EVLI) problems

Let X be a Banch space and X^* be its dual. Given a non-empty closed convex subset K of X and let $F:K\to X^*$, $\eta:K\times K\to X$ be two function. A proper preinvex lower semi continuous function $f:X\to\mathbb{R}\cup\{+\infty\}$, find $x_0\in K$ such that

$$\langle F(x_0),\eta(x,x_0)\rangle\geq f(x_0)-f(x), \text{ for all } x\in K$$

where $\langle\cdot,\cdot\rangle$ be the continuous dual paring on $X\times X^*$.

Now we introduce an extra function $\Omega:K\times K\to\mathbb{R}$. Let $\Omega(x,y)$ be non-negative, and for each $x\in K$, $\Omega(x,\cdot)$ is continuously differentiable on K. We further assume that $\Omega(x,x)=0$ and $\nabla_y\Omega(x,x)=0$ for all $x\in K$. Here $\nabla_y\Omega(x,x)$ denotes the Gâteaux derivative on $\Omega(x,y)$ with respect to the second variable, evaluated at $y=x$.

Our main goal is to show that the following function $\varphi:K\to\mathbb{R}$, where $K\subset X$, be convex subset, is ***gap function*** for $EVLI(3.1)$

$$\varphi(x)=\sup_{y\in K}\{\langle F(x),\eta(x,y)\rangle+f(x)-f(y)+\Omega(x,y)\}$$

Theorem (3.1) Let X ba areal Banach space and X^* be its dual space. Let K be a non-empty closed convex subset of X, $F:K\to X^*$ be a function and $f:X\to\mathbb{R}$ be a invex continuous function with respect to η, where $\eta(x,x)=0$, for all $x\in K$, satisfying condition C. Let $\Omega:K\times K\to\mathbb{R}$ be non-negative, and for each $x\in K$, $\Omega(x,\cdot)$ is continuously Gâteaux differentiable on K. Assume futher that for each $x\in K$, $\Omega(x,x)=0$ and $\nabla_y\Omega(x,x)=0$. Then φ (3.2), is a gap function for $EVLI(3.1)$.

Proof: Since $\Omega(x,x)=0$ and $\eta(x,x)=0$ for each $x\in K$, it is clear that $\varphi(x)\geq0$ for each $x\in K$. Now assume that x_0 solves $EVLI(3.1)$. Then

$$\langle F(x_0),\eta(x,x_0)\rangle\geq f(x_0)-f(x), \text{ for all } x\in K,$$

as $\Omega(x_0,x)\geq0$ for all $x\in K$,

$$\langle F(x_0), \eta(x, x_0)\rangle \geq f(x_0) - f(x) - \Omega(x_0, x), \text{ for all } x \in K$$

and $\varphi(x_0) \leq 0$. Hence $\varphi(x_0) = 0$.

Conversely, assume that $\varphi(x_0) = 0$. This implies that

$$\langle F(x_0), \eta(x, x_0)\rangle + f(x) - f(x_0) + \Omega(x_0, x) \leq 0, \text{ for all } x \in K$$

Define a mapping

$$x \to g(x) = \langle F(x_0), \eta(x, x_0)\rangle + f(x) + \Omega(x_0, x) \text{ for all } x \in K$$

Thus x_0 is a solution of the optimixation problem:

$$\text{minimize} \quad g(x) \quad \text{subject to } x \in K .$$

By the Proposition (2.6) and Remark (2), g is continuously Gâteaux differentiable at x_0, and hence locally Lipschtiz at x_0 and $\nabla g(x) \in \partial_l^\eta g(x)$.

Therefore, by Proposition (2.4), $\partial_l^\eta g(x_0)$ coincides with the subdifferential of g at x_0 in the sense of convex analysis and by the Proposition (3.1) and Proposition (3.3),

$$0 \in \partial_l^\eta g(x_0) + N_K^\eta(x_0) .$$

Hence by the Proposition (3.2),

$$0 \in F(x_0) + \partial_l^\eta f(x_0) + \nabla_x \Omega(x_0, x_0) + N_K^\eta(x_0)$$

Since $\nabla_x \Omega(x_0, x_0) = 0$, there exists an $v \in \partial_l^\eta f(x_0)$ such that $-F(x_0) - v \in N_K^\eta(x_0)$. Hence

$$\langle -F(x_0) - v, \eta(x, x_0)\rangle \leq 0 \text{ for all } x \in K,$$

$$\Rightarrow$$

$$\langle F(x_0), \eta(x, x_0)\rangle + \langle v, \eta(x, x_0)\rangle \geq 0 \text{ for all } x \in K,$$

Because f is pre-invex and $v \in \partial_l^\eta f(x_0)$, we have

$$f(x) - f(x_0) \geq \langle v, \eta(x, x_0)\rangle \text{ for all } x \in K \qquad (3.5)$$

From (3.4) and (3.5),

$$\langle F(x_0), \eta(x, x_0)\rangle + f(x) - f(x_0) \geq 0 \text{ for all } x \in K.$$

Therefore x_0 solves $EVLI$ (3.1). Hence the result proved.

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